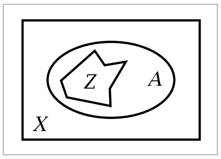
For example, taking B to be a point, the long exact sequence of the triple (X,A,B) becomes the long exact sequence of reduced homology for the pair (X,A).

Excision

A fundamental property of relative homology groups is given by the following **Excision Theorem**, describing when the relative groups $H_n(X,A)$ are unaffected by deleting, or excising, a subset $Z \subset A$.

Theorem 2.20. *Given subspaces $Z \subset A \subset X$ such that the closure of Z is contained in the interior of A, then the inclusion $(X-Z, A-Z) \hookrightarrow (X,A)$ induces isomorphisms $H_n(X-Z,A-Z) \to H_n(X,A)$ for all n. Equivalently, for subspaces $A,B \subset X$ whose interiors cover X, the inclusion $(B, A\cap B) \hookrightarrow (X,A)$ induces isomorphisms $H_n(B,A\cap B) \to H_n(X,A)$ for all n.*



The translation between the two versions is obtained by setting $B = X - Z$ and $Z = X - B$. Then $A \cap B = A - Z$ and the condition $\mathrm{cl}\, Z \subset \mathrm{int}\, A$ is equivalent to $X = \mathrm{int}\, A \cup \mathrm{int}\, B$ since $X - \mathrm{int}\, B = \mathrm{cl}\, Z$.

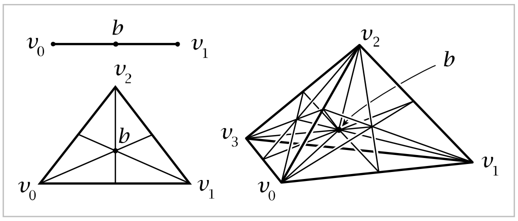
The proof of the excision theorem will involve a rather lengthy technical detour involving a construction known as barycentric subdivision, which allows homology groups to be computed using small singular simplices. In a metric space 'smallness' can be defined in terms of diameters, but for general spaces it will be defined in terms of covers.

For a space X, let $\mathcal{U} = \{U_j\}$ be a collection of subspaces of X whose interiors form an open cover of X, and let $C_n^{\mathcal{U}}(X)$ be the subgroup of $C_n(X)$ consisting of chains $\sum_i n_i\sigma_i$ such that each σ_i has image contained in some set in the cover $\mathcal{U}$. The boundary map $\partial : C_n(X) \to C_{n-1}(X)$ takes $C_n^{\mathcal{U}}(X)$ to $C_{n-1}^{\mathcal{U}}(X)$, so the groups $C_n^{\mathcal{U}}(X)$ form a chain complex. We denote the homology groups of this chain complex by $H_n^{\mathcal{U}}(X)$.

Proposition 2.21. *The inclusion $\iota : C_n^{\mathcal{U}}(X) \hookrightarrow C_n(X)$ is a chain homotopy equivalence, that is, there is a chain map $\rho : C_n(X) \to C_n^{\mathcal{U}}(X)$ such that $\iota\rho$ and $\rho\iota$ are chain homotopic to the identity. Hence ι induces isomorphisms $H_n^{\mathcal{U}}(X) \approx H_n(X)$ for all n.*

Proof: The barycentric subdivision process will be performed at four levels, beginning with the most geometric and becoming increasingly algebraic.

(1) *Barycentric Subdivision of Simplices.* The points of a simplex $[v_0, \cdots, v_n]$ are the linear combinations $\sum_i t_i v_i$ with $\sum_i t_i = 1$ and $t_i \ge 0$ for each i. The **barycenter** or 'center of gravity' of the simplex $[v_0, \cdots, v_n]$ is the point $b = \sum_i t_i v_i$ whose barycentric coordinates t_i are all equal, namely $t_i = 1/(n+1)$ for each i. The **barycentric subdivision** of $[v_0, \cdots, v_n]$ is the decomposition of $[v_0, \cdots, v_n]$ into the n-simplices $[b, w_0, \cdots, w_{n-1}]$ where, inductively, $[w_0, \cdots, w_{n-1}]$ is an $(n-1)$-simplex in the barycentric subdivision of a face $[v_0,\cdots,\hat{v}_i,\cdots,v_n]$. The induction starts with the case $n=0$ when the barycentric subdivision of $[v_0]$ is defined to be just $[v_0]$ itself. The next two cases $n=1,2$ and part of the case $n=3$ are shown in the figure. It follows from the inductive definition that the vertices of simplices in the barycentric subdivision of $[v_0,\cdots,v_n]$ are exactly the barycenters of all the k-dimensional faces $[v_{i_0},\cdots,v_{i_k}]$ of $[v_0,\cdots,v_n]$ for $0\le k\le n$. When $k=0$ this gives the original vertices v_i since the barycenter of a 0-simplex is itself. The barycenter of $[v_{i_0},\cdots,v_{i_k}]$ has barycentric coordinates $t_i=1/(k+1)$ for $i=i_0,\cdots,i_k$ and $t_i=0$ otherwise.

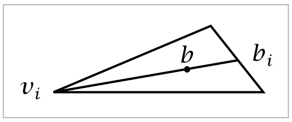


The n-simplices of the barycentric subdivision of Δ^n, together with all their faces, do in fact form a Δ-complex structure on Δ^n, indeed a simplicial complex structure, though we shall not need to know this in what follows.

A fact we will need is that the diameter of each simplex of the barycentric subdivision of $[v_0,\cdots,v_n]$ is at most $n/(n+1)$ times the diameter of $[v_0,\cdots,v_n]$. Here the diameter of a simplex is by definition the maximum distance between any two of its points, and we are using the metric from the ambient Euclidean space $\mathbb{R}^m$ containing $[v_0,\cdots,v_n]$. The diameter of a simplex equals the maximum distance between any of its vertices because the distance between two points v and $\sum_i t_iv_i$ of $[v_0,\cdots,v_n]$ satisfies the inequality

$$\big|v-\textstyle\sum_i t_iv_i\big| = \big|\sum_i t_i(v-v_i)\big| \le \sum_i t_i|v-v_i| \le \sum_i t_i\max|v-v_i| = \max|v-v_i|$$

To obtain the bound $n/(n+1)$ on the ratio of diameters, we therefore need to verify that the distance between any two vertices w_j and w_k of a simplex $[w_0,\cdots,w_n]$ of the barycentric subdivision of $[v_0,\cdots,v_n]$ is at most $n/(n+1)$ times the diameter of $[v_0,\cdots,v_n]$. If neither w_i nor w_j is the barycenter b of $[v_0,\cdots,v_n]$, then these two points lie in a proper face of $[v_0,\cdots,v_n]$ and we are done by induction on n. So we may suppose w_j, say, is the barycenter b, and then by the previous displayed inequality we may take w_k to be a vertex v_i. Let b_i be the barycenter of $[v_0,\cdots,\hat{v}_i,\cdots,v_n]$, with all barycentric coordinates equal to $1/n$ except for $t_i=0$. Then we have $b=\frac{1}{n+1}v_i+\frac{n}{n+1}b_i$. The sum of the two coefficients is 1, so b lies on the line segment $[v_i,b_i]$ from v_i to b_i, and the distance from b to v_i is $n/(n+1)$ times the length of $[v_i,b_i]$. Hence the distance from b to v_i is bounded by $n/(n+1)$ times the diameter of $[v_0,\cdots,v_n]$.



The significance of the factor $n/(n+1)$ is that by repeated barycentric subdivision we can produce simplices of arbitrarily small diameter since $(n/(n+1))^r$ approaches

0 as r goes to infinity. It is important that the bound $n/(n+1)$ does not depend on the shape of the simplex since repeated barycentric subdivision produces simplices of many different shapes.

(2) *Barycentric Subdivision of Linear Chains.* The main part of the proof will be to construct a subdivision operator $S:C_n(X)\to C_n(X)$ and show this is chain homotopic to the identity map. First we will construct S and the chain homotopy in a more restricted linear setting.

For a convex set Y in some Euclidean space, the linear maps $\Delta^n\to Y$ generate a subgroup of $C_n(Y)$ that we denote $LC_n(Y)$, the *linear chains*. The boundary map $\partial:C_n(Y)\to C_{n-1}(Y)$ takes $LC_n(Y)$ to $LC_{n-1}(Y)$, so the linear chains form a subcomplex of the singular chain complex of Y. We can uniquely designate a linear map $\lambda:\Delta^n\to Y$ by $[w_0,\cdots,w_n]$ where w_i is the image under λ of the i^{th} vertex of Δ^n. To avoid having to make exceptions for 0-simplices it will be convenient to augment the complex $LC(Y)$ by setting $LC_{-1}(Y)=\mathbb{Z}$ generated by the empty simplex $[\varnothing]$, with $\partial[w_0]=[\varnothing]$ for all 0-simplices $[w_0]$.

Each point $b\in Y$ determines a homomorphism $b:LC_n(Y)\to LC_{n+1}(Y)$ defined on basis elements by $b([w_0,\cdots,w_n])=[b,w_0,\cdots,w_n]$. Geometrically, the homomorphism b can be regarded as a cone operator, sending a linear chain to the cone having the linear chain as the base of the cone and the point b as the tip of the cone. Applying the usual formula for ∂, we obtain the relation $\partial b([w_0,\cdots,w_n])=[w_0,\cdots,w_n]-b(\partial[w_0,\cdots,w_n])$. By linearity it follows that $\partial b(\alpha)=\alpha-b(\partial\alpha)$ for all $\alpha\in LC_n(Y)$. This expresses algebraically the geometric fact that the boundary of a cone consists of its base together with the cone on the boundary of its base. The relation $\partial b(\alpha)=\alpha-b(\partial\alpha)$ can be rewritten as $\partial b+b\partial=\mathbb{1}$, so b is a chain homotopy between the identity map and the zero map on the augmented chain complex $LC(Y)$.

Now we define a subdivision homomorphism $S:LC_n(Y)\to LC_n(Y)$ by induction on n. Let $\lambda:\Delta^n\to Y$ be a generator of $LC_n(Y)$ and let b_λ be the image of the barycenter of Δ^n under λ. Then the inductive formula for S is $S(\lambda)=b_\lambda(S\partial\lambda)$ where $b_\lambda:LC_{n-1}(Y)\to LC_n(Y)$ is the cone operator defined in the preceding paragraph. The induction starts with $S([\varnothing])=[\varnothing]$, so S is the identity on $LC_{-1}(Y)$. It is also the identity on $LC_0(Y)$, since when $n=0$ the formula for S becomes $S([w_0])=w_0(S\partial[w_0])=w_0(S([\varnothing]))=w_0([\varnothing])=[w_0]$. When λ is an embedding, with image a genuine n-simplex $[w_0,\cdots,w_n]$, then $S(\lambda)$ is the sum of the n-simplices in the barycentric subdivision of $[w_0,\cdots,w_n]$, with certain signs that could be computed explicitly. This is apparent by comparing the inductive definition of S with the inductive definition of the barycentric subdivision of a simplex.

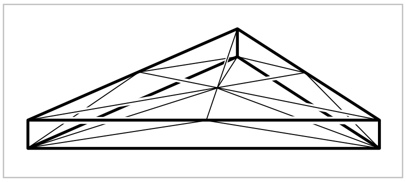
Let us check that the maps S satisfy $\partial S=S\partial$, and hence give a chain map from the chain complex $LC(Y)$ to itself. Since $S=\mathbb{1}$ on $LC_0(Y)$ and $LC_{-1}(Y)$, we certainly have $\partial S=S\partial$ on $LC_0(Y)$. The result for larger n is given by the following calculation, in which we omit some parentheses to unclutter the formulas:

$$
\begin{aligned}
\partial S\lambda &= \partial(b_\lambda(S\partial\lambda)) \\
&= S\partial\lambda - b_\lambda(\partial S\partial\lambda) \qquad \text{since } \partial b_\lambda + b_\lambda\partial = \mathbb{1} \\
&= S\partial\lambda - b_\lambda(S\partial\partial\lambda) \qquad \text{by induction on } n \\
&= S\partial\lambda \qquad \text{since } \partial\partial = 0
\end{aligned}
$$

We next build a chain homotopy $T: LC_n(Y) \to LC_{n+1}(Y)$ between S and the identity, fitting into a diagram

$$
\begin{array}{ccccccccc}
\cdots \longrightarrow & LC_2(Y) & \longrightarrow & LC_1(Y) & \longrightarrow & LC_0(Y) & \longrightarrow & LC_{-1}(Y) & \longrightarrow 0 \\
& \downarrow S & \overset{T}{\swarrow} & \downarrow S & \overset{T}{\swarrow} & S \downarrow \mathbb{1} & \overset{T}{\underset{0}{\swarrow}} & S \downarrow \mathbb{1} & \\
\cdots \longrightarrow & LC_2(Y) & \longrightarrow & LC_1(Y) & \longrightarrow & LC_0(Y) & \longrightarrow & LC_{-1}(Y) & \longrightarrow 0
\end{array}
$$

We define T on $LC_n(Y)$ inductively by setting $T = 0$ for $n = -1$ and letting $T\lambda = b_\lambda(\lambda - T\partial\lambda)$ for $n \ge 0$. The geometric motivation for this formula is an inductively defined subdivision of $\Delta^n \times I$ obtained by joining all simplices in $\Delta^n \times \{0\} \cup \partial\Delta^n \times I$ to the barycenter of $\Delta^n \times \{1\}$, as indicated in the figure in the case $n = 2$. What T actually does is take the image of this subdivision under the projection $\Delta^n \times I \to \Delta^n$.



The chain homotopy formula $\partial T + T\partial = \mathbb{1} - S$ is trivial on $LC_{-1}(Y)$ where $T = 0$ and $S = \mathbb{1}$. Verifying the formula on $LC_n(Y)$ with $n \ge 0$ is done by the calculation

$$
\begin{aligned}
\partial T\lambda &= \partial(b_\lambda(\lambda - T\partial\lambda)) \\
&= \lambda - T\partial\lambda - b_\lambda(\partial(\lambda - T\partial\lambda)) \qquad \text{since } \partial b_\lambda = \mathbb{1} - b_\lambda\partial \\
&= \lambda - T\partial\lambda - b_\lambda(S\partial\lambda + T\partial\partial\lambda) \qquad \text{by induction on } n \\
&= \lambda - T\partial\lambda - S\lambda \qquad \text{since } \partial\partial = 0 \text{ and } S\lambda = b_\lambda(S\partial\lambda)
\end{aligned}
$$

Now we are done with inductive arguments and we can discard the group $LC_{-1}(Y)$ which was used only as a convenience. The relation $\partial T + T\partial = \mathbb{1} - S$ still holds without $LC_{-1}(Y)$ since T was zero on $LC_{-1}(Y)$.

(3) *Barycentric Subdivision of General Chains.* Define $S: C_n(X) \to C_n(X)$ by setting $S\sigma = \sigma_\sharp S\Delta^n$ for a singular n-simplex $\sigma: \Delta^n \to X$. Since $S\Delta^n$ is the sum of the n-simplices in the barycentric subdivision of Δ^n, with certain signs, $S\sigma$ is the corresponding signed sum of the restrictions of σ to the n-simplices of the barycentric subdivision of Δ^n. The operator S is a chain map since

$$
\begin{aligned}
\partial S\sigma &= \partial\sigma_\sharp S\Delta^n = \sigma_\sharp \partial S\Delta^n = \sigma_\sharp S\partial\Delta^n \\
&= \sigma_\sharp S\left(\textstyle\sum_i (-1)^i \Delta_i^n\right) \qquad \text{where } \Delta_i^n \text{ is the } i^{th} \text{ face of } \Delta^n \\
&= \textstyle\sum_i (-1)^i \sigma_\sharp S\Delta_i^n \\
&= \textstyle\sum_i (-1)^i S(\sigma|\Delta_i^n) \\
&= S\left(\textstyle\sum_i (-1)^i \sigma|\Delta_i^n\right) = S(\partial\sigma)
\end{aligned}
$$

In similar fashion we define $T:C_n(X)\to C_{n+1}(X)$ by $T\sigma = \sigma_\sharp T\Delta^n$, and this gives a chain homotopy between S and the identity, since the formula $\partial T + T\partial = \mathbb{1} - S$ holds by the calculation

$$\begin{aligned}\partial T\sigma = \partial\sigma_\sharp T\Delta^n = \sigma_\sharp \partial T\Delta^n = \sigma_\sharp(\Delta^n - S\Delta^n - T\partial\Delta^n) &= \sigma - S\sigma - \sigma_\sharp T\partial\Delta^n \\ &= \sigma - S\sigma - T(\partial\sigma)\end{aligned}$$

where the last equality follows just as in the previous displayed calculation, with S replaced by T.

(4) *Iterated Barycentric Subdivision.* A chain homotopy between $\mathbb{1}$ and the iterate S^m is given by the operator $D_m = \sum_{0\le i<m} TS^i$ since

$$\begin{aligned}\partial D_m + D_m\partial = \sum_{0\le i<m}(\partial TS^i + TS^i\partial) = \sum_{0\le i<m}(\partial TS^i + T\partial S^i) = \\ \sum_{0\le i<m}(\partial T + T\partial)S^i = \sum_{0\le i<m}(\mathbb{1} - S)S^i = \sum_{0\le i<m}(S^i - S^{i+1}) = \mathbb{1} - S^m\end{aligned}$$

For each singular n-simplex $\sigma:\Delta^n\to X$ there exists an m such that $S^m(\sigma)$ lies in $C_n^{\mathcal{U}}(X)$ since the diameter of the simplices of $S^m(\Delta^n)$ will be less than a Lebesgue number of the cover of Δ^n by the open sets $\sigma^{-1}(\operatorname{int} U_j)$ if m is large enough. (Recall that a Lebesgue number for an open cover of a compact metric space is a number $\varepsilon > 0$ such that every set of diameter less than ε lies in some set of the cover; such a number exists by an elementary compactness argument.) We cannot expect the same number m to work for all σ's, so let us define $m(\sigma)$ to be the smallest m such that $S^m\sigma$ is in $C_n^{\mathcal{U}}(X)$.

Suppose we define $D:C_n(X)\to C_{n+1}(X)$ by $D\sigma = D_{m(\sigma)}\sigma$. To see whether D is a chain homotopy, we manipulate the chain homotopy equation

$$\partial D_{m(\sigma)}\sigma + D_{m(\sigma)}\partial\sigma = \sigma - S^{m(\sigma)}\sigma$$

into an equation whose left side is $\partial D\sigma + D\partial\sigma$ by moving the second term on the left side to the other side of the equation and adding $D\partial\sigma$ to both sides:

$$\partial D\sigma + D\partial\sigma = \sigma - \left[S^{m(\sigma)}\sigma + D_{m(\sigma)}(\partial\sigma) - D(\partial\sigma)\right]$$

If we define $\rho(\sigma)$ to be the expression in brackets in this last equation, then this equation has the form

$$(*) \qquad \partial D\sigma + D\partial\sigma = \sigma - \rho(\sigma)$$

We claim that $\rho(\sigma)\in C_n^{\mathcal{U}}(X)$. This is obvious for the term $S^{m(\sigma)}\sigma$. For the remaining part $D_{m(\sigma)}(\partial\sigma) - D(\partial\sigma)$, note first that if σ_j denotes the restriction of σ to the j^{th} face of Δ^n, then $m(\sigma_j)\le m(\sigma)$, so every term $TS^i(\sigma_j)$ in $D(\partial\sigma)$ will be a term in $D_{m(\sigma)}(\partial\sigma)$. Thus $D_{m(\sigma)}(\partial\sigma) - D(\partial\sigma)$ is a sum of terms $TS^i(\sigma_j)$ with $i\ge m(\sigma_j)$, and these terms lie in $C_n^{\mathcal{U}}(X)$ since T takes $C_{n-1}^{\mathcal{U}}(X)$ to $C_n^{\mathcal{U}}(X)$

We can thus regard the equation $(*)$ as defining $\rho:C_n(X)\to C_n^{\mathcal{U}}(X)$. For varying n these ρ's form a chain map since $(*)$ implies $\partial\rho(\sigma) = \partial\sigma - \partial D\partial(\sigma) = \rho(\partial\sigma)$.

The equation $(*)$ says that $\partial D + D\partial = \mathbb{1} - \iota\rho$ for $\iota : C_n^{\mathcal{U}}(X) \hookrightarrow C_n(X)$ the inclusion. Furthermore, $\rho\iota = \mathbb{1}$ since D is identically zero on $C_n^{\mathcal{U}}(X)$, as $m(\sigma) = 0$ if σ is in $C_n^{\mathcal{U}}(X)$, hence the summation defining $D\sigma$ is empty. Thus we have shown that ρ is a chain homotopy inverse for ι. □

Proof of the Excision Theorem: We prove the second version, involving a decomposition $X = A \cup B$. For the cover $\mathcal{U} = \{A, B\}$ we introduce the suggestive notation $C_n(A + B)$ for $C_n^{\mathcal{U}}(X)$, the sums of chains in A and chains in B. At the end of the preceding proof we had formulas $\partial D + D\partial = \mathbb{1} - \iota\rho$ and $\rho\iota = \mathbb{1}$. All the maps appearing in these formulas take chains in A to chains in A, so they induce quotient maps when we factor out chains in A. These quotient maps automatically satisfy the same two formulas, so the inclusion $C_n(A + B)/C_n(A) \hookrightarrow C_n(X)/C_n(A)$ induces an isomorphism on homology. The map $C_n(B)/C_n(A \cap B) \to C_n(A + B)/C_n(A)$ induced by inclusion is obviously an isomorphism since both quotient groups are free with basis the singular n-simplices in B that do not lie in A. Hence we obtain the desired isomorphism $H_n(B, A \cap B) \approx H_n(X, A)$ induced by inclusion. □

All that remains in the proof of Theorem 2.13 is to replace relative homology groups with absolute homology groups. This is achieved by the following result.

Proposition 2.22. *For good pairs (X, A), the quotient map $q : (X, A) \to (X/A, A/A)$ induces isomorphisms $q_* : H_n(X, A) \to H_n(X/A, A/A) \approx \widetilde{H}_n(X/A)$ for all n.*

Proof: Let V be a neighborhood of A in X that deformation retracts onto A. We have a commutative diagram

$$\begin{array}{ccccc}
H_n(X,A) & \longrightarrow & H_n(X,V) & \longleftarrow & H_n(X-A,V-A) \\
\downarrow{\scriptstyle q_*} & & \downarrow{\scriptstyle q_*} & & \downarrow{\scriptstyle q_*} \\
H_n(X/A,A/A) & \longrightarrow & H_n(X/A,V/A) & \longleftarrow & H_n(X/A-A/A,V/A-A/A)
\end{array}$$

The upper left horizontal map is an isomorphism since in the long exact sequence of the triple (X, V, A) the groups $H_n(V, A)$ are zero for all n, because a deformation retraction of V onto A gives a homotopy equivalence of pairs $(V, A) \simeq (A, A)$, and $H_n(A, A) = 0$. The deformation retraction of V onto A induces a deformation retraction of V/A onto A/A, so the same argument shows that the lower left horizontal map is an isomorphism as well. The other two horizontal maps are isomorphisms directly from excision. The right-hand vertical map q_* is an isomorphism since q restricts to a homeomorphism on the complement of A. From the commutativity of the diagram it follows that the left-hand q_* is an isomorphism. □

This proposition shows that relative homology can be expressed as reduced absolute homology in the case of good pairs (X, A), but in fact there is a way of doing this for arbitrary pairs. Consider the space $X \cup CA$ where CA is the cone $(A \times I)/(A \times \{0\})$